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Theorems for asymptotic safety of gauge theories

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Abstract We classify the weakly interacting fixed points of general gauge theories coupled to matter and explain how the competition between gauge and matter fluctuations gives rise to a rich spectrum of high- and low-energy fixed points. The pivotal role played by Yukawa couplings is emphasised. Necessary and sufficient conditions for asymptotic safety of gauge theories are also derived, in conjunction with strict no go theorems. Implications for phase diagrams of gauge theories and physics beyond the Standard Model are indicated.

1. Fixed points of the renormalisation group play an important role in quantum field theory and particle physics [1, 2]. Low-energy fixed points characterise continuous phase transitions and the dynamical breaking of symmetry. High-energy fixed points are central for the fundamental definition of quantum field theory. Important examples are provided by asymptotic freedom of non-abelian gauge theories [3, 4] where the high-energy fixed point is non-interacting. Gauge theories with complete asymptotic freedom, meaning asymptotic freedom for all of its couplings, are of particular interest in the search for extensions of the Standard Model [5]. Asymptotically free gauge theories can also display weakly coupled infrared (IR) fixed points [6, 7]. More recently, it was discovered that gauge theories can develop interacting ultraviolet (UV) fixed points [8], a scenario known as asymptotic safety. This intriguing new phenomenon, originally conjectured in the context of quantum gravity [9], offers the prospect for consistent UV completions of particle physics beyond the paradigm of asymptotic freedom [10].

We shall classify all weakly interacting fixed points of general gauge theories coupled to matter in four space-time

dimensions starting from first principles. Our motivation for doing so is twofold: firstly, we want to understand in general terms whether and how the competition between gauge and matter field fluctuations gives rise to quantum scale invariance. We expect that insights into conformal windows of gauge theories will offer new directions for particle physics above the electroweak energy scale. Secondly, we are particularly interested in the dynamical origin for asymptotic safety in gauge theories and conditions under which it may arise. We also hope that insights into the inner working of asymptotic safety at weak coupling will offer clues for mechanisms of asymptotic safety at strong coupling [11, 12].

We pursue these questions in perturbation theory starting with pure gauge interactions and gradually adding in more gauge and matter couplings. We will find a rich spectrum of interacting high- and low-energy fixed points including necessary and sufficient conditions for their existence. Furthermore, we highlight the central importance of Yukawa couplings to balance gauge against matter fluctuations. We thereby also establish that the presence of scalar fields such as the Higgs are strict necessary conditions for asymptotic safety at weak coupling. Further key ingredients for our results are bounds on quadratic Casimir operators which are derived for general Lie algebras, together with structural aspects of perturbation theory which are detailed as we proceed.

2. We begin our investigation of weakly coupled fixed points by considering (non-)abelian vector gauge theories with a simple gauge group \mathcal{G} and gauge coupling g , interacting with spin- $\frac{1}{2}$ fermions or scalars or both. Throughout we scale loop factors into the definition of couplings and introduce $\alpha = g^2/(4\pi)^2$. The renormalisation group running of the gauge coupling up to two-loop order in perturbation theory reads

$$\beta = -B\alpha^2 + C\alpha^3 + \mathcal{O}(\alpha^4), \quad (1)$$

where $\beta \equiv d\alpha/d(\ln \mu)$, and μ denoting the RG momentum scale. The one- and two-loop coefficients in (1) are known

In section 11, line 11 the figure numbers were incorrect. It should read: "Phase diagrams of simple $4d$ gauge theories with $N = 1$ supersymmetry take the form Figs. 2, 3 and 4 while settings with Fig. 5 cannot be realised."

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for arbitrary field content and given in [3, 4, 6, 13–17], respectively. In terms of the Dynkin index S_2^R and the quadratic Casimir operator C_2^R of quantum fields in some irreducible representation R of the gauge group, they can be written as¹

$$B = \frac{2}{3} \left(11C_2^G - 2S_2^F - \frac{1}{2}S_2^S \right), \quad (2)$$

$$C = 2 \left[\left(\frac{10}{3}C_2^G + 2C_2^F \right) S_2^F + \left(\frac{1}{3}C_2^G + 2C_2^S \right) S_2^S - \frac{34}{3}(C_2^G)^2 \right]. \quad (3)$$

The terms involving C_2^G —the quadratic Casimir operator in the adjoint representation of the gauge group—arise due to the fluctuations of the gauge fields. The fluctuations of charged fermionic (F) or scalar (S) matter fields, if present, contribute to (1) via the terms proportional to the Dynkin index of their representation.

Gauge theories with (1) will always display the free Gaussian fixed point $\alpha_* = 0$. If $B > 0$ this is the well-known ultraviolet (UV) fixed point of asymptotic freedom [3, 4] such as in QCD. For $B < 0$, instead, the theory becomes free in the infrared (IR) such as in QED. In addition, (1) can also display an interacting fixed point

$$\alpha_* = \frac{B}{C} \quad (4)$$

which is perturbative if $\alpha_* \ll 1$ and physically acceptable provided that $B \cdot C > 0$. For $B \cdot C < 0$ the would-be fixed point reads $\alpha_* < 0$ and resides in an unphysical regime where the theory is sick non-perturbatively [18]. Also, if $B < 0$ ($B > 0$), (4) corresponds to an interacting UV (IR) fixed point. We conclude that the availability and nature of interacting fixed points is encoded in the signs and magnitude of (2) and (3). From the explicit expressions, we observe that the pure gauge contributions to both the one- and two-loop terms are either negative (non-abelian) or vanishing (abelian). Conversely, terms originating from fermionic or scalar matter contribute positively. This means that with a sufficiently small amount of matter (including none), the gauge boson contributions dominate and we have $B > 0$, $C < 0$. On the other hand, for a sufficiently large amount of matter, the matter contributions dominate and we end up with $B \leq 0$, $C > 0$. The latter is trivially the case for abelian gauge groups whose quadratic Casimir operator vanishes identically, $C_2^{U(1)} = 0$. Weakly interacting fixed points are absent in either of these cases.

The question of what may happen when the pure gauge and matter contributions are of similar size is not immediately obvious. It has long been known that it is possible for theories to have $B, C > 0$, which are therefore asymptotically free and which, if $B \ll C$, can lead to a perturbative infrared

Banks–Zaks fixed point [6, 7]. However, no examples have been found for which $B, C < 0$ and where the analogous fixed point would be ultraviolet. To see if such a scenario is possible in principle, we must examine the relative effects of matter on the one- and two-loop contributions. To that end, we resolve (2) for the adjoint Casimir operator and insert the result into the last term of (3) to find

$$C = \frac{2}{11} [2S_2^F (11C_2^F + 7C_2^G) + 2S_2^S (11C_2^S - C_2^G) - 17B C_2^G]. \quad (5)$$

We make the following observations. The first term in (5) due to the fermions is manifestly positive-definite. The last term in (5) is positive-definite provided that $B < 0$. Hence, as has been noted by Caswell [6], fermionic matter alone cannot generate an asymptotically safe UV fixed point in perturbation theory. The middle term, however, due to charged scalars, is not manifestly positive-definite and it cannot be decided *prima facie* whether or not it may generate an interacting UV fixed point with $B < 0$ and $C < 0$.

3. In order to proceed with the analysis of (5), we must find expressions for the smallest quadratic Casimir operator for any simple Lie algebra \mathcal{G} . Irreducible representations of simple Lie algebras are conveniently characterised by their highest weight Λ , which for a rank- n Lie algebra is an n -dimensional vector of non-negative integers, not all of which are zero.² This is due to the theorem of highest weight, which states that inequivalent irreducible representations are in one-to-one correspondence with distinct highest weights. The Racah formula offers an explicit expression for the quadratic Casimir operator for any irreducible representation R with highest weight Λ . It is given by

$$C_2(\Lambda) = \frac{1}{2}(\Lambda, \Lambda + 2\delta), \quad (6)$$

where $(u, v) \equiv \sum_{ij} G_{ij} u^i v^j$ denotes the inner product of two highest weights, with $u = \sum_{i=1}^n u^i \Lambda_i$. The weight metrics $G \equiv (G_{ij})$ are known explicitly for any Lie algebra \mathcal{G} . Note that $(u, v) > 0$ for any two weights. The n -component vector δ in (6) denotes half the sum of the positive roots and reads $\delta = (1, 1, \dots, 1)$ in the Dynkin basis (which we use exclusively). The normalisation factor $\frac{1}{2}$ in (6) is conventional.³

For any Lie algebra, the highest weight of irreducible representations with the smallest quadratic Casimir operator must be one of the fundamental weights Λ_k (with $k \in \{1, \dots, n\}$), whose components are defined as

² We are not interested in trivial representations given that uncharged fields cannot contribute to (1).

³ In general, the quadratic Casimir operator is only defined up to a multiplicative constant for a given Lie algebra, and thus we are free to choose the overall normalisation.

¹ Throughout, we treat fermions as Weyl and scalars as real.

$$(\Lambda_k)^i = \delta_k^i. \quad (7)$$

This can be understood as follows. Consider two highest weights Λ and λ , which may be used to construct a new irreducible representation with highest weight $\Lambda + \lambda$. The bilinearity of the inner product (6) then implies that

$$C_2(\Lambda + \lambda) > C_2(\Lambda) + C_2(\lambda) > C_2(\Lambda). \quad (8)$$

It follows, trivially, that C_2 can be made arbitrarily large. To find the smallest C_2 , however, (8) states that we only need to consider irreducible representations whose highest weights have a single non-vanishing component. Assuming Λ to be one such weight and taking $\lambda = m\Lambda$ for some integer $m \geq 1$, (8) also states that we only need to consider highest weights where this single non-vanishing component takes the smallest non-vanishing value, which is unity. This establishes (7). Inserting (7) into (6), and denoting by G the weight metric of the gauge group \mathcal{G} , we find the quadratic Casimir operator in terms of the fixed index k as

$$C_2 = \frac{1}{2} G_{kk} + \sum_{i=1}^n G_{ki}. \quad (9)$$

It remains to identify the minima of (9) with respect to k for the four classical and the five exceptional Lie algebras separately, following the Cartan classification, starting with the rank- n classical Lie algebras A_n , B_n , C_n and D_n [19]. For $n \geq 1, 2, 3$ and 4 they correspond to the unique Lie algebras $\mathfrak{su}(n+1)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$ and $\mathfrak{so}(2n)$, respectively. Explicit expressions for the weight metrics are summarised in [20]. For our purposes we write them in closed form as

$$\begin{aligned} (G^{A_n})_{ij} &= \min(i, j) - \frac{ij}{n+1}, \\ (G^{B_n})_{ij} &= \frac{1}{2} \left[\min(i, j)(2 - \delta_{in} - \delta_{jn}) + \frac{n}{2} \delta_{in} \delta_{jn} \right], \\ (G^{C_n})_{ij} &= \frac{1}{2} \min(i, j), \\ (G^{D_n})_{ij} &= \frac{1}{2} \left[\min(i, j)(2 - \delta_{in} - \delta_{jn} - \delta_{i,n-1} - \delta_{j,n-1}) \right. \\ &\quad \left. + \frac{n}{2} (\delta_{i,n-1} \delta_{j,n-1} + \delta_{in} \delta_{jn}) \right. \\ &\quad \left. + \frac{1}{2} (n-2)(\delta_{i,n-1} \delta_{j,n} + \delta_{i,n} \delta_{j,n-1}) \right]. \end{aligned} \quad (10)$$

For illustration, we consider explicitly the case for A_n , where $G_{kk} = k(n+1-k)/(n+1)$, which, combined with

$$\begin{aligned} \sum_{i=1}^n G_{ki} &= \sum_{i=1}^k i + \sum_{i=k+1}^n k - \frac{k}{n+1} \sum_{i=1}^n i \\ &= \frac{1}{2} k (n+1-k), \end{aligned}$$

leads to the desired expression for $C_2(A_n)$ as stated in (11) below. Analogous, if slightly more tedious, intermediate steps for the other cases lead to the result

$$\begin{aligned} C_2(A_n) &= \frac{k}{2} \frac{(n+1-k)(n+2)}{n+1}, \\ C_2(B_n) &= \frac{1}{2} \left(k(2n+1-k) - \frac{1}{4} n(3+2n) \delta_{kn} \right), \\ C_2(C_n) &= \frac{k}{2} \left(n+1 - \frac{1}{2} k \right), \\ C_2(D_n) &= \frac{1}{2} \left(k(2n-k) - \frac{n}{4} (2n-3+4k) (\delta_{k,n-1} + \delta_{kn}) \right), \end{aligned} \quad (11)$$

with k taking values between 1 and n . To find the global minima of the expressions (11) with respect to k , we proceed as follows. For A_n and C_n , the expressions are quadratic polynomials in k with negative k^2 coefficient, implying that its minima are achieved at the boundaries, meaning either $k = 1$ or $k = n$, or both. For B_n and D_n , additionally, the expressions are discontinuous for certain intermediate values of k (owing to the $\delta_{k,n-1}$ and δ_{kn} factors). This implies that global minima may additionally be achieved for integer values of k within the interval $(1, n)$. With this in mind, and after evaluating all possible cases, the final result for the smallest quadratic Casimir operator for the classical Lie algebras is found to be

$$\begin{aligned} \min C_2(A_n) &= \frac{n}{2} \frac{n+2}{n+1}, \\ \min C_2(B_n) &= \begin{cases} \frac{1}{8} n(2n+1) & \text{for } n = 2, 3 \\ n & \text{for } n \geq 4 \end{cases}, \\ \min C_2(C_n) &= \frac{n}{2} + \frac{1}{4}, \\ \min C_2(D_n) &= n - \frac{1}{2}. \end{aligned} \quad (12)$$

The five exceptional groups $E_{6,7,8}$, F_4 , and G_2 have a fixed size, hence finding the smallest Casimir operator amounts to a simple minimisation. Using the appropriate expressions for the weight metrics [20], our results are summarised in Table 1 where, for convenience, we express (12) using the particle physics nomenclature for the gauge groups.

A few comments are in order: (1) For A_n either boundary is minimal, corresponding to the fundamental and anti-fundamental representation. (2) For B_n the Casimir operator is minimal for $k = n$ (the fundamental spinor representation) provided $n = 2$ or 3, and for $k = 1$ (the fundamental vector representation) provided $n \geq 4$. (3) For C_n and D_n , the Casimir operator is minimal for $k = 1$ (the fundamental vector representation). (4) For D_4 , three smallest Casimir operators are achieved for $k = 1, 3$ and 4. This degeneracy is due to the fact that the Dynkin diagram for D_4 possesses a three-fold symmetry, and thus there

Table 1 Summary of minimal Casimir operators for the classical and exceptional Lie algebras along with the Casimir operator in the adjoint representation, their ratio χ , and the representations that attain the min-

imum. We notice that for D_4 , corresponding to $SO(8)$, the Dynkin diagram has a three-fold symmetry leading to triality amongst the smallest Casimir operators in the fundamental vector and spinor representations

Symmetry	Range	Min C_2	C_2 (adj)	χ	Irrep with smallest C_2
$SU(N)$	$N \geq 2$	$\frac{N^2-1}{2N}$	N	$\frac{1}{2} \left(1 - \frac{1}{N^2}\right)$	Fundamental N and \bar{N}
$SO(N)$	$3 \leq N \leq 7$	$\frac{1}{16}N(N-1)$	$N-2$	$\frac{N}{16} \frac{N-1}{N-2}$	Fundamental spinors $2^{\lceil N/2 \rceil - 1}$
	$N = 8$	$\frac{7}{2}$	6	$\frac{7}{12}$	Fundamental vector 8_v and fundamental spinors $8_s, 8_c$
	$N \geq 9$	$\frac{1}{2}(N-1)$	$N-2$	$\frac{N-1}{2(N-2)}$	Fundamental N
$Sp(N)$	$N \geq 1$	$\frac{1}{4}(2N+1)$	$N+1$	$\frac{2N+1}{4(N+1)}$	Fundamental $2N$
E_8		30	30	1	Adjoint 248
E_7		$\frac{57}{4}$	18	$\frac{19}{24}$	Fundamental 56
E_6		$\frac{26}{3}$	12	$\frac{13}{18}$	Fundamental 27 and $\bar{27}$
F_4		6	9	$\frac{2}{3}$	Fundamental 26
G_2		2	4	$\frac{1}{2}$	Fundamental 7

is a triality between the fundamental vector and the two inequivalent spinor representations. (5) For the exceptional groups, we find that the smallest Casimir operator is unique, except for E_6 . (6) E_8 is the only group where the smallest Casimir operator is achieved for the adjoint representation (which is also one of the fundamental representations). (7) While the quadratic Casimir operator in general is a non-monotonic function of the dimensionality of the representation, our findings establish that the smallest Casimir operator always corresponds to those representations with the smallest dimension, which is always one of the fundamental representations.

Since the overall normalisation of quadratic Casimir operators (6) can be chosen freely, it is useful to consider the ratio between the smallest quadratic Casimir operator and the Casimir operator in the adjoint representation,

$$\chi = \frac{\min C_2(R)}{C_2(\text{adj})}, \quad (13)$$

which is independent of the normalisation. Figure 1 shows our results for χ for all simple Lie algebras. Evidently, χ is going to be bounded from above $\chi \leq 1$ because the adjoint representation always exists. The upper boundary is achieved for the exceptional group E_8 . Furthermore, χ is also bounded from below,

$$\frac{3}{8} \leq \chi \leq 1. \quad (14)$$

The lower bound is achieved for the fundamental two-dimensional representation of $SU(2) \simeq SO(3) \simeq Sp(1)$, and for the two inequivalent two-dimensional representation of $SO(4)$. We observe that χ is an increasing function with

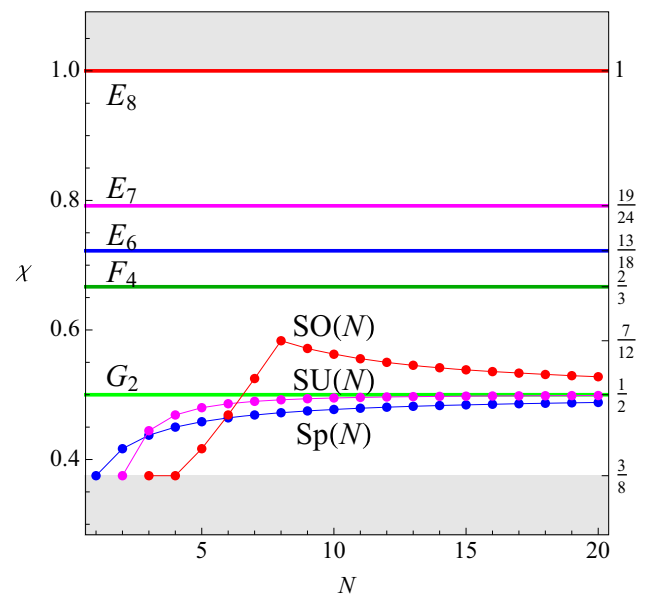


Fig. 1 Shown is the ratio χ , see (13)—the smallest achievable quadratic Casimir operator in units of the Casimir operator in the adjoint representation—for all simple Lie algebras. The grey areas show the excluded domains. We observe that $\frac{3}{8} \leq \chi \leq 1$. The lower bound is achieved for the fundamental two-dimensional representation of $SU(2) \simeq SO(3) \simeq Sp(1)$, and for the two inequivalent two-dimensional representation of $SO(4)$. For the exceptional groups the smallest Casimir operator grows with the rank of the group. The upper bound is achieved for E_8 . In all cases, the smallest quadratic Casimir operator is achieved for the irreducible representation of smallest dimensionality

N for $SU(N)$ and $Sp(N)$, interpolating between $\frac{3}{8}$ for small N and $\frac{1}{2}$ in the infinite- N limit. For $SO(N)$, we find that χ grows from $\frac{3}{8}$ to its maximum $\frac{7}{12}$ at $N = 8$, from where it

decays with increasing N towards $\frac{1}{2}$ from above. From the exceptional groups, only G_2 has a χ value close to those of the classical groups. All other exceptional groups have larger values for χ , which furthermore increases with the rank of the group.

4. We are now in a position to develop the central results of this work, summarised in Tables 2 and 3. We have observed in (5) that charged scalars potentially may turn the two-loop coefficient C negative even if $B \leq 0$, provided that non-trivial scalar irreducible representations are found with $C_2^S < \frac{1}{11} C_2^G$. However, the result (13), (14) now firmly establishes that this is out of reach for any simple Lie algebra, owing to $C_2^S \geq \frac{3}{8} C_2^G$. Moreover, we find that the two-loop coefficient obeys

$$C \geq C_2^G \left(\frac{89}{22} S_2^F + \frac{25}{22} S_2^S - \frac{34}{11} B \right) \quad (15)$$

for any non-abelian gauge theory. Hence, while it is possible to have B parametrically small such as in a Veneziano limit with suitably rescaled gauge coupling [21], the result (15) also shows that it is impossible to have both B and C parametrically small. Most importantly, we conclude that, for any gauge theory with a vanishing or positive one-loop coefficient for its gauge coupling's β function, the two-loop coefficient is necessarily positive,

$$B \leq 0 \Rightarrow C > 0; \quad (16)$$

see (1). It is worth noting that (16) is not an equivalence: while $C < 0$ arises exclusively only if $B > 0$, the case $C > 0$ can arise irrespective of the sign of B [6, 7]. Consequently, Banks–Zaks fixed points are invariably IR fixed points. From the viewpoint of the asymptotic safety conjecture, our result (16) has the form of a no go theorem: within perturbation theory, irrespective of the matter content and in the absence of non gauge interactions, asymptotic safety cannot be realised for any four-dimensional simple non-abelian, or abelian, gauge theory.⁴

The result (16) straightforwardly generalises to matter fields in generic reducible representations under the gauge symmetry. In this case it suffices to replace terms involving Dynkin indices and matter Casimir operators in the one- and two-loop coefficients by

$$S_2^R \rightarrow \sum_i S_2^{R_i}, \quad S_2^R C_2^R \rightarrow \sum_i S_2^{R_i} C_2^{R_i}, \quad (17)$$

where the sums run over the decomposition into irreducible representations of the fermionic ($R = F$) and scalar ($R = S$) matter fields. Applying (17) to the two-loop coefficient (5),

⁴ Caswell has observed some time back that “We do not expect to find a gauge theory of the above type [meaning with (1)] where β starts out positive and goes negative near enough to the origin for the zero to be valid in perturbation theory.” [6]. Our result (16) offers a general proof for Caswell’s conjecture.

we find that all fermionic contributions remain manifestly positive-definite, and that each summand of the scalar contributions is positive-definite owing to (13), (14). We conclude that the no go theorem (16) holds true for general matter representations, as summarised in Table 2b.

5. Turning to more general gauge interactions, we consider gauge theories with product gauge groups $\mathcal{G} \equiv \otimes_{a=1}^n \mathcal{G}_a$ and multiple gauge couplings α_a , each associated with a simple or abelian factor \mathcal{G}_a . We assume the presence of scalar and/or fermionic matter fields, some or all of which are charged under some or all of the gauge symmetries. In the absence of Yukawa interactions, the β functions for the gauge couplings up to two loops in perturbation theory are of the form

$$\beta_a = \alpha_a^2 (-B_a + C_{ab} \alpha_b) + \mathcal{O}(\alpha^4), \quad (18)$$

and $a, b = 1, \dots, n$. The coefficients B_a and C_{aa} (no sum) are the standard one- and two-loop coefficients of the gauge coupling α_a as given in (2), (3). The new terms at two-loop level are the off-diagonal contributions C_{ab} ($a \neq b$), which parametrise the $\mathcal{O}(\alpha_b)$ contributions to the renormalisation group flow of couplings α_a . Non-trivial mixing between two gauge couplings arises through matter fields which are charged under both of these. The mixing terms can then be written as [22, 23]

$$C_{ab} = 4(C_2^{F_b} S_2^{F_a} + C_2^{S_b} S_2^{S_a}) \quad (a \neq b). \quad (19)$$

The subscripts a, b on the Casimir operator or Dynkin index of the matter fields indicate the subgroup of \mathcal{G} . From (19) it follows that the mixing terms are manifestly non-negative ($C_{ab} \geq 0$) for any semi simple quantum gauge theory with or without abelian factors. Equation (19) has a straightforward generalisation for reducible representations. Furthermore, if the theory contains more than one abelian factor, the off-diagonal contributions take a slightly different form in the presence of kinetic mixing [24, 25]. In either of these cases, the mixing terms remain manifestly non-negative ($C_{ab} \geq 0$, $a \neq b$). Together with (16) for all diagonal entries, we find that

$$B_a \leq 0 \Rightarrow C_{ab} \geq 0 \quad \text{for all } b, \quad (20)$$

meaning that, for every infrared free gauge group factor \mathcal{G}_a , the corresponding column of the two-loop gauge contribution matrix (C_{ab}) is non-negative.

The result (20) has immediate implications for interacting fixed points of quantum field theories with (18), which, to leading order in perturbation theory, are given by all solutions of the linear equations

$$B_a = C_{ab} \alpha_b^*, \quad \text{subject to } \alpha_b^* \geq 0. \quad (21)$$

Assuming that $B_a \leq 0$ for at least one of the subgroups \mathcal{G}_a , it follows from (20) that, for (21) to have a solution, at least one of the fixed points α_b^* must take negative values. However, we

have already explained that such solutions are inconsistent [18], and conclude that the theory cannot have physically acceptable interacting fixed points within the perturbative regime as soon as any of the gauge factors is infrared free ($B_a \leq 0$). In other words, the result (20) has the form of a no go theorem: asymptotic safety cannot be achieved for any semi-simple quantum gauge theory of the type (18) with or without abelian factors and irrespective of the matter content.

Reversing the line of reasoning, our findings also establish that physically acceptable interacting fixed points in gauge theories with (18) and without Yukawa interactions can only be achieved if all gauge group factors are asymptotically free ($B_a > 0$), which excludes $U(1)$ factors straightaway; see Table 3b. All weakly interacting fixed point solutions of (21) are necessarily IR fixed points of the Banks–Zaks type inasmuch as they arise from balancing one- and two-loop gauge field fluctuations. They also display a lesser number of relevant directions than the asymptotically free Gaussian UV fixed point meaning that UV–IR connecting trajectories exist which flow from the Gaussian down to any of the interacting fixed points.

Next, we investigate scalar and Yukawa-type matter couplings, and clarify whether these may help to generate weakly interacting fixed points.

6. Scalar self-interactions arise unavoidably in settings with charged scalars owing to the fluctuations of the gauge fields or in settings with uncharged scalars as long as these couple indirectly to the gauge fields through charged fermions and Yukawa interactions. Quartic scalar self-interactions or cubic ones in a phase with spontaneous symmetry breaking renormalise the gauge couplings starting at the three-loop (four-loop) level in perturbation theory, provided the scalars are charged (uncharged) [26].

In the light of (16), to help generate an interacting fixed point in the gauge sector once $B \leq 0$, the scalar couplings would have to outweigh the one-loop as well as the two-loop gauge contributions. Even if the one-loop term vanishes identically ($B = 0$), the result (14) together with (2), (5) and (15) establishes that the two-loop gauge coefficient is strictly positive $C(B = 0) \geq C_{\min}$ and of order unity, with

$$C_{\min}/(C^G)^2 = 22\frac{1}{4}. \quad (22)$$

The absolute minimum (22) is achieved for $Sp(1)$, $SU(2)$, $SO(3)$ and $SO(4)$ gauge symmetries. The bound becomes slightly stronger with increasing N , reaching $C_{\min}/(C^G)^2 = 25$ for the classical Lie groups in the infinite N limit. For the exceptional groups G_2 , F_4 , E_6 , E_7 and E_8 we find the increasingly stronger bounds $C_{\min}/(C^G)^2 = 25, 50\frac{2}{3}, 55\frac{5}{9}, 61\frac{2}{3}$ and 80, respectively. Notice also that for all gauge groups the minimum is achieved for charged fermions only. The presence of charged scalars systematically enhances $C > C_{\min}$. Thus, coming back to the scalar self-interactions,

even in the most favourable scenario where the one-loop coefficient vanishes and the gauge coupling is perturbatively small, a cancellation between the two-loop gauge and the three- or four-loop scalar contributions requires scalar couplings of order unity owing to the lower bounds (15), (22).⁵ Hence, the feasibility of such a scenario necessitates non-perturbatively large scalar couplings, outside the perturbative domain. We conclude that non-abelian gauge theories with any type of self-interacting scalar matter, and with or without fermionic matter but without Yukawa interactions, cannot become asymptotically safe within perturbation theory. This result also completes the no go theorems stated in Table 2b, c in the presence of scalar matter.

7. Yukawa couplings are naturally present in settings with both scalar and fermionic matter fields [27,28], and contribute to the running of (some of) the gauge couplings provided that (some of) the fermions carry charges under (some of) the gauge groups. Scalars may or may not carry charges. Yukawa couplings are technically natural [29] and cannot be switched-on by fluctuations: the limit of vanishing Yukawa couplings constitutes an exact fixed point of the theory.

For concreteness we consider simple non-abelian or abelian gauge theories with the most general Yukawa interactions taking the form $\sim \frac{1}{2}(\mathbf{Y}^A)_{JL}\phi^A\psi_J\zeta\psi_L$ with $\zeta = \pm i\sigma_2$, with Weyl indices suppressed. In perturbation theory the Yukawa couplings \mathbf{Y}^A contribute to the renormalisation of the gauge coupling starting at the two-loop level, and the beta function (1) is replaced by [30]

$$\beta = \alpha^2(-B + C\alpha - 2Y_4). \quad (23)$$

The Yukawa couplings enter through the new term $Y_4 = \text{Tr}[\mathbf{C}_2^F \mathbf{Y}^A (\mathbf{Y}^A)^\dagger]/d(G)$, with $d(G)$ the dimension of the gauge group, \mathbf{Y}^A the (matrix of) Yukawa couplings, \mathbf{C}_2^F the matrix of quadratic Casimir operators of the fermionic irreducible representations, and the trace summing over all fermionic indices. Notice that we have scaled the loop factor of (4π) into the definition of \mathbf{Y}^A . The coefficients B and C are as in (2) and (3). In general, the matrix \mathbf{C}_2^F is diagonal according to the fermionic irreducible representations, implying that Y_4 is positive as long as (some of) the Yukawa couplings are non-vanishing. Positivity of Y_4 can be made manifest by rewriting it as

$$Y_4 = \sum_{AJL} S_2^{FJ} |(\mathbf{Y}^A)_{JL}|^2/d(F_J) \geq 0. \quad (24)$$

It follows that Yukawa couplings contribute with an overall negative sign to the running of gauge couplings, irrespective of the sign of the one-loop gauge coefficient B . Assuming that the Yukawa couplings, and thus Y_4 , take a fixed point of their own, interacting fixed points of (23) take the form

⁵ For this estimate we have assumed that the relevant loop factor $(4\pi)^2$ is scaled into the definition of the scalar self-coupling, consistent with our conventions for the gauge and Yukawa couplings.

Table 2 Asymptotic safety in gauge theories coupled to matter with a–c stating strict no go theorems and d–e necessary and sufficient conditions

Case	Gauge group	Matter	Yukawa	Asymptotic safety	Info
a	Simple	Fermions in irreducible representations	No	No	Reference [6]
b	Simple or abelian	Fermions, any rep	No	No	(16)
		Scalars, any rep	No	No	(16), (22)
		Fermions and scalars, any rep	No	No	(16), (22)
c	Semi-simple, with or without abelian factors	Fermions, any rep	No	No	(20)
		Scalars, any rep	No	No	(20), (22)
		Fermions and scalars, any rep	No	No	(20), (22)
d	Simple or abelian	Fermions and scalars, any rep	Yes	Yes	(31), (38)
e	Semi-simple, with or without abelian factors	Fermions and scalars, any rep	Yes	Yes	(21), (38)

(4) except that the one-loop coefficient is effectively shifted $B \rightarrow B' = B + 2Y_4^*$, with

$$B' \geq B. \quad (25)$$

This Yukawa-induced shift has important implications. Most notably, in settings where the gauge sector is asymptotically non-free ($B \leq 0$), the Yukawa contribution Y_4^* may effectively change the sign of the one-loop coefficient ($B' > 0$), thereby enabling a viable interacting fixed point

$$\alpha_* = \frac{B'}{C}. \quad (26)$$

In more physical terms, for infrared free theories these findings state that the growth of the gauge coupling with energy, as dictated by the positive one- and two-loop gauge contributions (16), is invariably slowed down, and, as long as $B' > 0$, eventually brought to a halt by Yukawa interactions. In particular, the occurrence of a UV Landau pole in the gauge coupling can be avoided dynamically. As we have shown earlier, neither scalar self-interactions nor further gauge couplings are able to negotiate a fixed point at weak coupling once $B \leq 0$. We therefore conclude that Yukawa interactions are the *only* type of interactions that can generate an interacting UV fixed point for *any* weakly coupled gauge theory.

In view of the above it is useful to investigate the Yukawa sector in more detail. To that end, we exploit the explicit flow for the Yukawa couplings $\beta^A = dY^A/d\ln\mu$. At the leading non-trivial order in perturbation theory which is one loop, it takes the form [31, 32]

$$\beta^A = \mathbf{E}^A(Y) - \alpha \mathbf{F}^A(Y). \quad (27)$$

The terms $\mathbf{E}^A(Y)$, which are of cubic order in the Yukawa couplings, arise from fluctuations of the fermion and scalar fields and encode vertex and propagator corrections [31]. General expressions for \mathbf{E}^A in the conventions adopted here are given in [33, 34]. The terms $\mathbf{F}^A(Y) = 3\{C_2^F, Y^A\}$ origi-

nate primarily from gauge field fluctuations and are (block-)diagonally proportional to Y^A following the fermion irreducible representations [32]. Scalar self-couplings contribute to (27) starting at two loop and can be neglected for sufficiently small couplings.

The nullcline condition $\beta^A(Y, \alpha) = 0$ for the Yukawa couplings has two types of solutions. The Gaussian fixed point $\mathbf{Y}_*^A = 0$ always exists, because both \mathbf{E}^A and \mathbf{F}^A vanish individually for vanishing Yukawa couplings, whence $\beta^A(Y = 0, \alpha) = 0$. In addition, and provided that the gauge coupling is non-vanishing, the two terms in (27) can balance against each other. Dimensional analysis shows that the functions $\bar{\beta}^A(C) \equiv \beta^A(\sqrt{\alpha}C, \alpha)/\alpha^{3/2}$ are independent of the gauge coupling α , implying that Yukawa nullclines take the form

$$\mathbf{Y}_*^A = \frac{g}{4\pi} \mathbf{C}^A. \quad (28)$$

The “reduced” Yukawa couplings \mathbf{C}^A are numerical matrices independent of the gauge coupling g which solve $\bar{\beta}^A(C) = 0$, meaning $\mathbf{E}^A(C) = \mathbf{F}^A(C)$ for $\mathbf{C}^A \neq 0$. Evidently $\mathbf{C}^A = 0$ corresponds to the Gaussian.⁶ The solutions (28) are promoted to genuine fixed points of the coupled system (23), (27) iff the gauge coupling simultaneously takes a real fixed point g_* (26). At the fixed point, perturbativity in the Yukawa couplings then follows parametrically from perturbativity in the gauge coupling.

Inserting the nullcline back into (23) we find that the Yukawa-induced terms are of order α^3 owing to (28). This establishes that the shifted one-loop coefficient B' depends linearly on α through Y_4^* , meaning that (26) constitutes an implicit equation for α_* . The implicit dependences are

⁶ For any nullcline \mathbf{C}^A (28), $-\mathbf{C}^A$ and $\mathbf{C}^{A\dagger} = \mathbf{C}^{A*}$ are physically equivalent nullclines. In the literature one-loop nullclines are sometimes referred to as “fixed points” (for the reduced couplings) or “eigenvalue conditions” [35].

Table 3 Summary of weakly interacting fixed points in gauge theories, detailing the availability of Banks–Zaks (BZ) or gauge–Yukawa (GY) type fixed points, or combinations and products thereof

Case	Gauge group	Yukawa	Parameter	Interacting FPs	Type	Info
a	Simple	No	$B > 0$ and $C > 0$	Banks–Zaks	IR	References [6, 7]
b	Semi-simple, no $U(1)$ factors	No	All $B_a > 0$	Banks–Zaks and products thereof	IR	Soln of (21)
c	Simple	Yes	$B > 0$ and $C > 0 > C'$	Banks–Zaks	IR	Figure 3
	Simple	Yes	$B > 0$ and $C > C' > 0$	BZ and GYs	IR	Figure 4
	Simple or abelian	Yes	$B < 0$ and $C' < 0$	Gauge–Yukawas	UV/IR	Figure 5
d	Semi-simple, with or without $U(1)$ factors	Yes	All $B'_a > 0$	BZs and GYs and products thereof	UV/IR	Soln of (21)

resolved by accounting for the Yukawa contributions as, effectively, modifications of the two-loop coefficient. We find

$$Y_4 = D \cdot \alpha \quad (29)$$

where the coefficient $D = \text{Tr}[\mathbf{C}_2^F \mathbf{C}^A (\mathbf{C}^A)^\dagger]/d(G) \geq 0$ only depends on group-theoretical weights and the reduced Yukawa couplings parametrising the nullcline, but not on the gauge coupling. The projection of the flow for the gauge coupling (23) along a hypersurface with $\beta^A = 0$ then takes the form (1) except that the two-loop gauge coefficient C is shifted into $C \rightarrow C' = C - 2D$. The shift term vanishes iff all Yukawa couplings vanish but is strictly negative otherwise, whence

$$C' \leq C. \quad (30)$$

This result makes it manifest that Yukawa contributions can dynamically lower the effective two-loop coefficient, possibly avoiding the no go theorem (16). Furthermore, the shift (30) implies that interacting fixed points for the gauge coupling take the form (4) with $C \rightarrow C'$,

$$\alpha_* = \frac{B}{C'}. \quad (31)$$

We stress that the expressions (26) and (31) for the gauge coupling fixed point are equivalent and numerically identical. For practical purposes, however, the latter representation, if available, is preferred as it provides the fully resolved version of the former. Following on from our earlier discussion, the fixed points (31) are physical as long as $B \cdot C' > 0$, and perturbative if $|B| \ll |C'|$. If $B > 0$ and $C' > 0$, they constitute infrared fixed points of the theory, similar to Banks–Zaks fixed points except for the additional presence of Yukawa interactions. If $B < 0$ and $C' < 0$, they constitute interacting UV fixed points and qualify as asymptotically safe UV completions for the theory; see Table 3c for a summary. No such weakly coupled UV completion can arise without Yukawa interactions.

We conclude that Yukawa couplings offer a dynamical mechanism to negotiate interacting fixed points in gauge theories. Most importantly, for asymptotically non-free gauge theories with $B \leq 0$, they offer a *unique* mechanism to generate weakly interacting fixed points. The strict no go theorem (16) may then be circumnavigated under the auxiliary condition that the Yukawa-induced shift term comes out large enough for C' to turn negative. This result, summarised in Table 2d, thus takes the form of a necessary condition for asymptotic safety.

8. Our results are straightforwardly generalised to gauge–Yukawa theories with several abelian or non-abelian gauge group factors, assuming that some or all of the fermions are charged under some or all of the gauge groups, while the scalars may or may not be charged. The renormalisation of the gauge couplings then takes the form [30]

$$\beta_a = \alpha_a^2 (-B_a + C_{ab} \alpha_b - 2Y_{4,a}), \quad (32)$$

where the two-loop Yukawa contributions now arise through $Y_{4,a} = \text{Tr}[\mathbf{C}_2^{F_a} \mathbf{Y}^A (\mathbf{Y}^A)^\dagger]/d(G_a) \geq 0$. As is evident from the explicit expression, the quadratic Casimir operator of the fermions takes the role of a projector to identify the contributions to the running of α_a . The running of the Yukawa couplings continues to be given by (27), except that further gauge field contributions turn the last term into a sum over gauge groups $\alpha \mathbf{F}^A \rightarrow \alpha_a \mathbf{F}_a^A$ with $\mathbf{F}_a^A(Y) = 3\{\mathbf{C}_2^{F_a}, \mathbf{Y}^A\}$ [33]. This modification leads to a larger variety of Yukawa nullclines, depending on which of the gauge couplings take vanishing or non-vanishing values at the fixed point. Provided that some or all of the Yukawa couplings take interacting fixed points they will contribute to the running of the gauge couplings (32) through $Y_{4,a}^* \geq 0$. Consequently, the gauge beta functions reduce to the form (18) except that the one-loop coefficients are effectively shifted, $B_a \rightarrow B'_a = B_a + 2Y_{4,a}^*$, due to the fixed point in the Yukawa sector. Most importantly, we observe that

$$B'_a \geq B_a. \quad (33)$$

Equality holds true iff all Yukawa couplings take Gaussian values. The shift (33) implies that gauge coupling fixed points of the theory arise as the solutions of

$$B'_a = C_{ab} \alpha_b^*, \quad \text{subject to } \alpha_b^* \geq 0. \quad (34)$$

Once more, this structure has important implications. Following on from our earlier discussion of (21), the fixed point condition (21) can have physical solutions iff all B'_a are positive. Due to (33) this is naturally the case as long as each gauge group factor is asymptotically free. The theory is then asymptotically free in all gauge factors with interacting fixed points of the Banks–Zaks and the gauge–Yukawa type, and combinations and products thereof. The decisive difference with (21) comes into its own for theories where some or all B_a are negative. Provided that the Yukawa-induced shift terms ensure that all B'_a become positive numbers even if one or several of the gauge factors are not asymptotically free, the fixed point condition (21) can have a variety of novel solutions; see Table 3d. Such fixed points are genuinely of the gauge–Yukawa type, and furthermore constitute candidates for asymptotically safe UV completions of the theory. Also, no such fixed point can arise out of theories with (21), which once more highlights the pivotal role played by Yukawa interactions.

As a final remark, we note that the fixed point condition (21) still depends implicitly on the gauge couplings through B'_a , once Y_4 is evaluated on a nullcline. It is straightforward to resolve the implicit dependence provided that $Y_{4,a}$ takes the form

$$Y_{4,a} = D_{ab} \alpha_b \quad (35)$$

along Yukawa nullclines, in analogy to (29).⁷ Continuity in each of the gauge couplings $\alpha_b \geq 0$ together with the non-negativity of $Y_{4,a}$ allows us to observe that the matrix (D_{ab}) is non-negative. The flow of the gauge couplings (32) is reduced to (18), except that the two-loop term is shifted $C_{ab} \rightarrow C'_{ab} = C_{ab} - 2 D_{ab}$ following (35). We conclude that the Yukawa contributions along nullclines effectively reduce the two-loop gauge contributions to the renormalisation of

gauge couplings. In this representation, the fixed point condition (21) turns into the equivalent form

$$B_a = C'_{ab} \alpha_b^*, \quad \text{subject to } \alpha_b^* \geq 0. \quad (36)$$

For non-negative C'_{ab} , as has been shown above, interacting fixed points can only be realised if all gauge group factors are asymptotically free. Here, however, the matrix (C'_{ab}) is no longer required to be strictly non-negative, unlike the matrix (C_{ab}) of two-loop gauge contributions, and the no go theorem (20) can be avoided owing to the Yukawa contributions. In view of the asymptotic safety conjecture, this completes our proof that charged fermions with charged or uncharged scalars and, most crucially, Yukawa interactions, constitute strictly necessary ingredients for interacting UV fixed points in general weakly coupled gauge theories; see Table 2e.

9. Gauge–Yukawa fixed points necessitate scalar fields. Consequently, two auxiliary conditions arise: firstly, the scalar sector must achieve a fixed point of its own, interacting or otherwise. Secondly, the scalar sector must admit a stable ground state. To appreciate that both of these requirements are non-empty, we consider the renormalisation group flow $\beta = d\lambda/d\ln\mu$ for the quartic scalar couplings $\lambda = (\lambda_{ABCD})$ based on the interaction Lagrangian $\sim \frac{1}{4!} \lambda_{ABCD} \phi^A \phi^B \phi^C \phi^D$. To leading order the beta functions $\beta = \beta(\lambda, Y, \alpha)$ depend quadratically on the quartics, on the Yukawa and gauge couplings, and on group-theoretical factors related to the gauge transformations of the scalars (if charged) [32]. Explicit expressions and generalisations for product gauge groups can be found in [34, 36]. Scalar self-couplings are not technically natural [29] and can be switched on by fluctuations of the fermions (due to the presence of Yukawa couplings) or by fluctuations of the gauge fields (if the scalars are charged), implying that $\beta(\lambda = 0, Y, \alpha) \neq 0$ in general.

Next we turn to the scalar nullclines $\beta = 0$, subject to $\beta^A \rightarrow 0$. Using dimensional analysis, we observe that the functions $\bar{\beta}(\bar{C}, C) \equiv \beta(\alpha \bar{C}, \alpha C, \alpha)/\alpha^2$ are α -independent. The implicit solutions \bar{C} of the quadratic algebraic equations $\bar{\beta}(\bar{C}(C), C) = 0$ provide us with

$$\lambda_* = \alpha \bar{C}. \quad (37)$$

The “reduced” scalar couplings \bar{C} are numerical tensors which depend on group-theoretical factors and the reduced Yukawa couplings, but not explicitly on the gauge coupling. Since the quartics do not impact on the gauge–Yukawa flow (to leading order) it is immaterial for this analysis whether the gauge coupling is slowly running or sitting on a fixed point.

Qualitatively and quantitatively different types of solutions λ_* arise for all physically inequivalent Yukawa nullclines with $C^A \neq 0$, and with $C^A \rightarrow 0$. In either of these cases, owing to the quadratic nature of the defining equations,

⁷ The form (35) is evident if only one of the gauge couplings, say g_b , is non-vanishing. The nullcline takes the form $\mathbf{Y}_{b,*}^A = \frac{g_b}{4\pi} \mathbf{C}_b^A$, see (28), with \mathbf{C}_b^A a solution of $\mathbf{E}^A(C) = \mathbf{F}_b^A(C)$, leading to $D_{ab} = \text{Tr}[\mathbf{C}_2^{F_a} \mathbf{C}_b^A (\mathbf{C}_b^A)^\dagger] / d(G_a) \geq 0$. More generally, (35) holds true for any quantum field theory whose one-loop Yukawa vertex corrections obey $\mathbf{Y}^B \mathbf{Y}^{\dagger A} \mathbf{Y}^B = \mathbf{Y}^A \text{Tr} \mathbf{M}^{BC} (\mathbf{Y}^{\dagger B} \mathbf{Y}^C + \mathbf{Y}^{\dagger C} \mathbf{Y}^B)$ for some matrix $(\mathbf{M}^{BC})_{JL} = m_J^B \delta^{BC} \delta_{JL}$, which is block-diagonally proportional to the identity in field space with real m_J^B . In these cases the flow for the Yukawa couplings (27) are mapped explicitly onto closed flows for their squares $|(\mathbf{Y}^A)_{JK}|^2$ whose nullclines, and consequently $Y_{4,a}$ on nullclines, are linear functions of the squares of the gauge couplings, α_b . In theories with more complex Yukawa vertex corrections (e.g. Pati–Salam, trinification) the relation between $Y_{4,a}$ and α_b takes a more general form.

solutions (37) generically come up in inequivalent pairs \bar{C}_{\pm} per Yukawa nullcline with complex entries. Reality of quartic couplings is not automatically guaranteed and must be required as an auxiliary condition. Vacuum stability necessitates that λ_* is a positive-definite tensor.⁸ This information is not encoded in the renormalisation group flow even if the scalar couplings come out real, meaning that the stability of the effective potential $V_{\text{eff}}(\phi)$ provides an independent constraint. We therefore conclude that (37), subject to

$$\lambda_{ABCD}^* = \text{real}, \quad \text{and} \quad V_{\text{eff}}(\phi) = \text{stable}, \quad (38)$$

are mandatory auxiliary conditions for gauge theories with scalar matter to display a physically acceptable scalar sector, in addition to the conditions for free or interacting fixed points in the gauge or gauge–Yukawa sectors.

A few comments are in order: (1) solutions of (38) with $C^A \neq 0$ are mandatory for gauge–Yukawa fixed points and for asymptotic safety [8,38]. Those with $C^A = 0$ are mandatory for Banks–Zaks fixed points in the presence of scalar matter. (2) Both equations of (38) must be imposed irrespective of the UV or IR nature of the underlying fixed point. (3) If two solutions \bar{C}_{\pm} are physical, one of them is UV and the other IR relevant. (4) Solutions to (37), (38) also control trajectories in the vicinity of free or interacting fixed points [35]. Those with $C^A \neq 0$ entail that gauge, Yukawa, and scalar couplings run at the same rate and govern the approach to gauge–Yukawa fixed points. Those with $C^A \rightarrow 0$ (referring to reduced Yukawa couplings which approach the Gaussian very rapidly $Y^A(\alpha)/\sqrt{\alpha} \equiv C^A(\alpha) \ll 1$) are relevant for asymptotically free theories to display complete asymptotic freedom, and for trajectories approaching Banks–Zaks fixed points. Scalar couplings then run into the Gaussian UV fixed point either alongside the gauge coupling, or faster $\lambda_*(\alpha)/\alpha \ll 1$. The latter follows from the α -dependence of the reduced Yukawa couplings $C^A(\alpha)$, which entails an implicit α -dependence for the quartics [32]. (4) A method to find solutions in the limit $C^A \rightarrow 0$ has been detailed in [39]. Physical solutions for the combined Yukawa and scalar nullclines with (38) exist and are known for a number of theories [40–43].⁹

This completes the derivation of necessary and sufficient conditions of existence for weakly interacting fixed points in general gauge theories coupled to matter.

10. Next, we return to the starting point of our investigation where we observed that the competition between gauge field and matter fluctuations, and hence the relative signs and size of the loop coefficient B and C (for theories with

a simple gauge group) determines the fixed point structure. However, it has become clear that a third quantity, C' , controlled by Yukawa interactions, plays an equally important role. To illustrate its impact, we turn to a brief discussion of weakly coupled gauge theories from the viewpoint of their phase diagrams. Four distinct cases arise: Besides the Gaussian fixed point, gauge theories either display none, the Banks–Zaks, gauge–Yukawa, or the Banks–Zaks and gauge–Yukawa fixed points, depending on the values for B , C , and C' ; see Table 3c. The different phase diagrams are shown qualitatively in Figs. 2, 3, 4 and 5, projected onto the (α, Y_4) plane.

Gauge theories with $B > 0$ and $C < 0$ have no weakly coupled fixed points. At weak coupling, the phase diagram solely displays asymptotic freedom and the Gaussian UV fixed point, Fig. 2. The set of UV free trajectories emanating out of it are indicated by the red-shaded area. Its upper boundary is provided by the Yukawa nullcline which also acts as an infrared attractor [45–49] due to the fact that the sign of (27) is always controlled by the gauge field fluctuations for small Yukawa couplings. On the scaling trajectory, the gauge, Yukawa and scalar couplings run at the same rate into the Gaussian UV fixed point [35]. UV free trajectories continue towards the domain of strong coupling where the theory is expected to display confinement and chiral symmetry breaking, or, possibly, a strongly coupled IR fixed point. On the other hand, above the Yukawa nullcline no trajectories

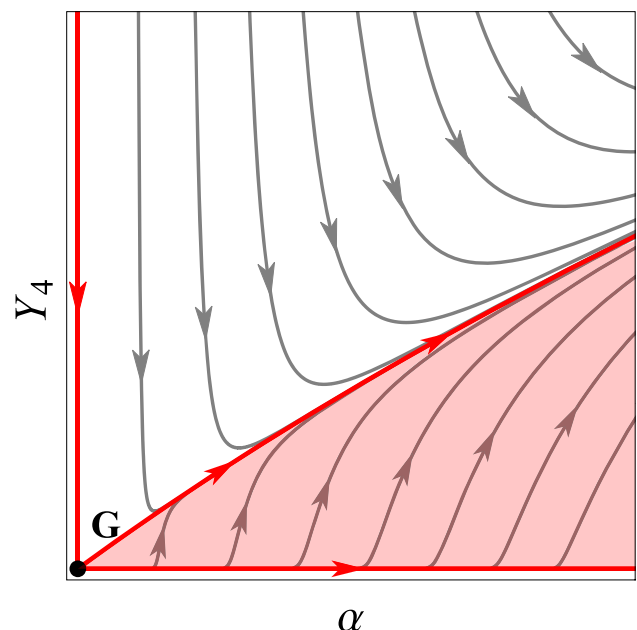


Fig. 2 Phase diagram of gauge–Yukawa theories with $B > 0$ and $C < 0$ at weak coupling showing asymptotic freedom and the Gaussian UV fixed point (G). Arrows indicate the flow towards the IR. The red-shaded area covers the set of UV complete trajectories emanating from the Gaussian UV fixed point. The Yukawa nullcline acts on trajectories as an IR attractor

⁸ In the presence of flat directions, Coleman–Weinberg type resummations [37] for the leading logarithmic corrections of the effective potential will have to be invoked [15].

⁹ See [5,8,38,44] for recent results in the context of asymptotic safety and asymptotic freedom, respectively.

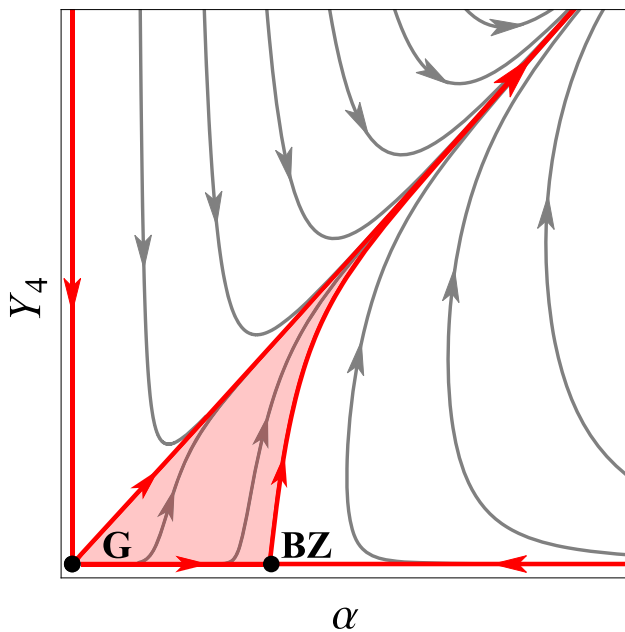


Fig. 3 Phase diagram of gauge–Yukawa theories with $B > 0$ and $C > 0 > C'$ at weak coupling showing asymptotic freedom with the Gaussian and the Banks–Zaks fixed point (BZ). Notice the funneling of all UV free trajectories towards the Yukawa nullcline as furthered by the Banks–Zaks fixed point

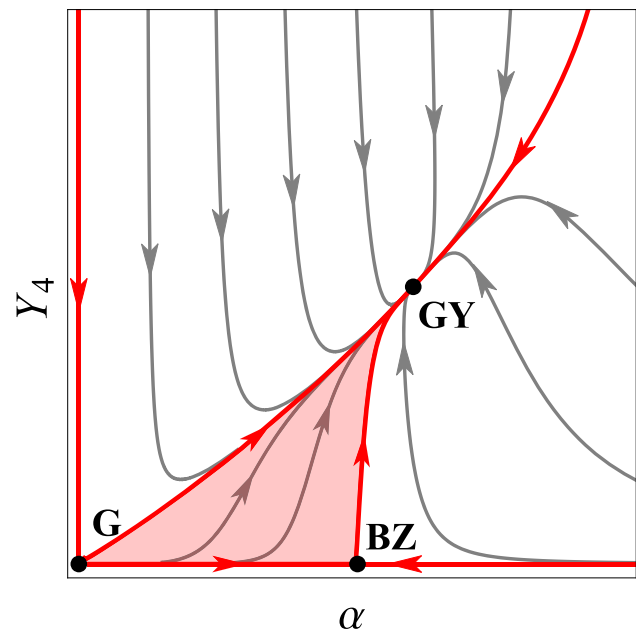


Fig. 4 Fixed points and phase diagrams of gauge–Yukawa theories with $B > 0$ and $C > C' > 0$ at weak coupling showing asymptotic freedom with Gaussian, Banks–Zaks, and gauge–Yukawa fixed points (GY). Notice that the gauge–Yukawa fixed point attracts UV free trajectories emanating from the Gaussian

are found which can reach the Gaussian in the UV. On such trajectories, the theory technically loses asymptotic freedom. Predictability is then limited up to a finite UV scale, unless a strongly coupled UV fixed point materialises out of the blue.

Gauge theories with $B > 0$ and $C > 0 > C'$ additionally develop a Banks–Zaks fixed point (4) which is perturbative provided B/C is sufficiently small. Yukawa couplings are immaterial for this. Banks–Zaks fixed points are always weakly attractive in the gauge and strongly repulsive in the Yukawa direction. The former follows from asymptotic freedom together with (23), while the latter follows from (27) and $\partial F^A / \partial Y^B$ being non-negative and proportional to the gauge coupling times the sum of the quadratic Casimir operators of the fermions attached to the vertex. Moreover, at weak coupling and close to the Banks–Zaks, the flow is always parametrically faster into the Y_4 than into the gauge direction. Consequently, the Banks–Zaks fixed point together with the Yukawa nullcline act as a strong infrared-attractive funnel for all trajectories emanating from the Gaussian UV fixed point; see Fig. 3. This leads to low-energy relations between the Yukawa and the gauge coupling dictated by (27) (at weak coupling), irrespective of their detailed UV origin.¹⁰ Otherwise the same discussion as in the previous example applies.

Progressing towards gauge theories with $B > 0$ and $C > C' > 0$ we now additionally observe a fully inter-

acting gauge–Yukawa fixed point besides the Banks–Zaks, displayed in Fig. 4. The main new effect in theories with $C' > 0$ as opposed to those with $C' < 0$ is that the funneling of flow trajectories towards the IR attractive Yukawa nullcline comes to a halt, whereby couplings take an interacting IR fixed point (28), (31). Furthermore, the fixed point is genuinely attractive in both the gauge and the Yukawa directions.¹¹ The theory comes out more strongly coupled at the gauge–Yukawa than at the Banks–Zaks fixed point owing to (30). The gauge–Yukawa fixed point characterises a second order phase transition between a symmetric phase and a phase with spontaneous symmetry breaking where the scalars acquire a non-vanishing vacuum expectation value. Details of the phase transition becomes visible once mass terms are added, taking the role of temperature, with the scalar vacuum expectation values serving as order parameters. Spontaneous symmetry breaking may also entail the breaking of chiral symmetry via Yukawa couplings. Away from fixed points, the theory may display a number of further phenomena such as first order phase transitions, dimensional transmutation, decoupling, and confinement in the deep IR.¹²

¹⁰ Exact examples are given by the gauge–Yukawa theories of [8] in the parameter range $0 < 11/2 - N_F/N_C \ll 1$.

¹¹ In theories with several Yukawa couplings several gauge–Yukawa fixed point may arise of which at least one is fully IR attractive. See [50] for an explicit example with a single Yukawa coupling.

¹² Phenomenological aspects of IR gauge–Yukawa fixed points have been pioneered in [50, 51] (see also [52, 53]). Models with gauge–

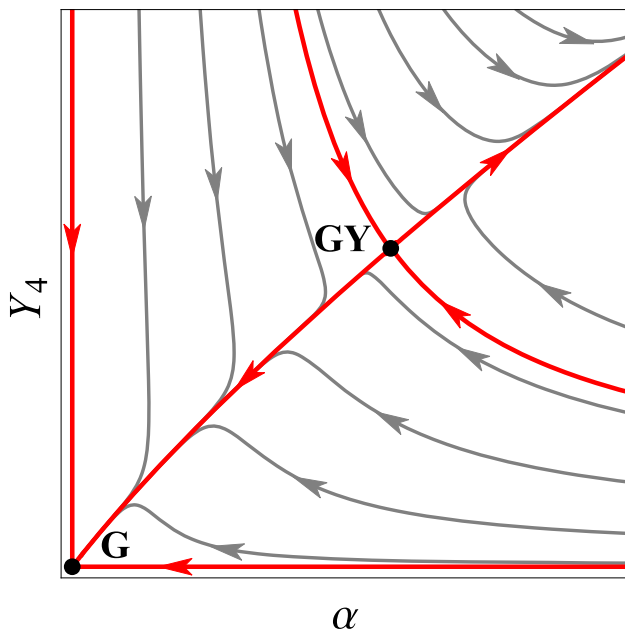


Fig. 5 Fixed points and phase diagrams of gauge–Yukawa theories with $B < 0$ and $C' < 0$ at weak coupling showing asymptotic safety together with the Gaussian and gauge–Yukawa fixed points. Notice that the set of UV finite trajectories is confined to a hypercritical surface dictated by the Yukawa nullcline

Turning to simple or abelian gauge theories with $B < 0$ and $C' < 0$ we observe that asymptotic freedom is absent and the Gaussian has become an infrared fixed point. Also, it is impossible for this type of theories to have a Banks–Zaks fixed point owing to the no go theorem (16). However, the Yukawa interactions have turned the two-loop coefficient $C > 0$ effectively into $C' < 0$ allowing for an interacting gauge–Yukawa fixed point (31) as displayed in Fig. 5. This fixed point genuinely displays an attractive and a repulsive direction, the former being a consequence of the IR attractive nature of Yukawa nullclines, and the latter a consequence of infrared freedom in the gauge coupling. Moreover, it qualifies as an asymptotically safe fixed point owing to the two UV finite trajectories emanating out of it [8]. The weak coupling trajectory connects the interacting fixed point with the Gaussian in the infrared whereby the theory remains unconfined at all scales. The strong coupling trajectory, as in the previous cases, is expected to lead to confinement and chiral symmetry breaking, or conformal behaviour at low energies. Away from the Yukawa nullcline (which always coincides with the hypercritical surface of the gauge–Yukawa fixed point), no trajectories are found which can reach the gauge–Yukawa fixed point in the UV. On such trajectories, the theory technically loses asymptotic safety and predictivity is limited by

a maximal UV scale unless a novel UV fixed point emerges at strong coupling.

As an aside, it is worth noticing a similarity between gauge–Yukawa theories with complete asymptotic freedom and a Banks–Zaks, and gauge–Yukawa theories with asymptotic safety; see Figs. 3 and 5. In both cases, trajectories which escape from the UV fixed point region towards strong coupling in the IR are solely determined by the Yukawa nullcline. All settings predict IR relations between Yukawa and gauge couplings. In the former case this arises due to a funnel effect while in the latter it follows from the unstable direction of the interacting UV fixed point. Without Banks–Zaks, IR relations may be avoided at the expense of substantial fine-tuning in the deep UV; see Fig. 2.

The discussion of phase diagrams generalises to more complex settings. Gauge theories with several independent Yukawa couplings will lead to several parameters C' , which, depending on their magnitudes, may generate several gauge–Yukawa fixed points. Phase diagrams will then display an enhanced structure owing to additional cross-over phenomena amongst the various fixed points. An even richer pattern arises for theories with product gauge groups, see Table 3d. Here, the gauge loop coefficients B_a and C_{ab} together with the Yukawa-induced coefficients B'_a uniquely determine the fixed point structure at weak coupling. Evidently, for each gauge coupling individually our discussion based on the “diagonal” coefficients B , C and C' applies, meaning that parts of the enlarged phase diagrams materialise as “direct products” of those shown in Figs. 2, 3, 4 and 5. As a novel addition, theories will also display “off-diagonal” Banks–Zaks and gauge–Yukawa fixed points as well as fully interacting products thereof, depending on the availability and structure of the solutions to (21).¹³ Furthermore, each interacting fixed point naturally relates to a conformal window similar to those of QCD with fermionic matter. Some of the fixed points of (product) gauge theories offer UV conformal windows around fixed points with exact asymptotic safety at weak coupling. It is therefore natural to speculate that some such models may qualify as UV completions for the Standard Model of particle physics.

11. Finally, we briefly comment on interacting fixed points in $4d$ supersymmetric QFTs. Supersymmetry imposes relations amongst gauge, Yukawa, and scalar couplings [57]. In general, quartic scalar self-interactions are no longer independent. For theories with $N = 1$ supersymmetry without superpotentials, gauge beta functions remain of the form (1) at weak coupling. The signs of B and C depend on the matter content [17]. Gauge sectors can develop Banks–Zaks fixed points (4) which are always IR ($B > 0$) but never UV [58], fully consistent with our findings in non-supersymmetric the-

Footnote 12 continued

Yukawa fixed points have also been studied from the viewpoint of conformal field theory [54] and the a theorem [55].

¹³ See [56] for a recent example in semi-simple gauge theories without Yukawa couplings.

ories (16), (20). An important difference arises once superpotentials (i.e. Yukawa couplings) are present. Owing to supersymmetry, Yukawas can only take weakly interacting fixed points provided at least one of the gauge sectors is asymptotically free [58]. This implies that asymptotic safety at weak coupling is out of reach for simple $N = 1$ supersymmetric gauge theories. Overall, weakly interacting fixed points are either absent, or of the Banks–Zaks, or of the gauge–Yukawa type. Phase diagrams of simple $4d$ gauge theories with $N = 1$ supersymmetry take the form Figs. 2, 3 and 4 while settings with Fig. 5 cannot be realised. For $N = 2$ supersymmetry, Yukawa couplings are no longer independent but related to the gauge coupling. Moreover, the running of the gauge coupling becomes one loop exact with (1) and $C \equiv 0$ [59,60]. Hence, $N = 2$ theories are either asymptotically free or infrared free and interacting fixed points cannot arise. In the limit where $B = 0$, the gauge coupling becomes exactly marginal leading to a line of fixed points [59]. The latter continues to hold true for maximally extended supersymmetry, $N = 4$ SYM, where the constraints from supersymmetry are so powerful that the theory does not flow under the RG, and any value of the gauge coupling corresponds to a fixed point.¹⁴

12. In summary, we have identified the interacting fixed points of four-dimensional gauge theories in the regime where gauge and matter fields remain good fundamental degrees of freedom. Low-energy fixed points are either of the Banks–Zaks or gauge–Yukawa type, or combinations and products thereof (Table 3), offering a rich spectrum of phenomena including phase transitions and the spontaneous breaking of symmetry. We have also derived no go theorems together with necessary and sufficient conditions to guarantee asymptotic safety of general gauge theories (Table 2). Interacting high-energy fixed points are invariably of the gauge–Yukawa type and require elementary scalar fields such as the Higgs. Hence, the findings of [8] were not a coincidence: rather, the dynamical mechanism to tame the notorious Landau poles of general infrared free gauge theories is *unique*, and, owing to the group-theoretical limitation (14), *exclusively* delivered through Yukawa interactions. We conclude that our findings open a window of opportunities towards perturbative UV completions of the Standard Model beyond the paradigm of asymptotic freedom.

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¹⁴ For further constraints on supersymmetric fixed points including at strong coupling, see [58,61].

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